ZERO-SUM FLOWS OF THE LINEAR LATTICE

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ABSTRACT. A zero-sum flow of a graph G is an element of the nullspace of the incidence matrix of G whose coefficients are nonzero real numbers. A zero-sum flow is called a k-flow if all the coefficients of the nullspace vector are integers less than k in absolute value. It is conjectured that any graph with a zero-sum flow must admit a 6-flow. In this note, we consider the lattice of subspaces of an n-dimensional vector space over a finite field. We prove the existence of zero-sum flows for the incidence matrix between two levels of the linear lattice with different rank numbers. Using field-theoretic considerations, we also show that there exists an $([m]_q + 1)$-flow or $([n-m]_q + 1)$-flow between levels 1 and m for $2 \le m \le n-2$ whenever m or $n-m$, respectively, divide n. Additionally, if neither m nor $n-m$ divide n, we show there exists a 2- or 3-flow between levels 1 and m.

1. INTRODUCTION

1.1. Motivation and Literature.

For a matrix M with real entries, a *zero-sum flow* is an element of the nullspace of M with no zero entries. A *k-flow* for the matrix M is a zero-sum flow with integer entries where the absolute value of each entry is less than k. In other words, $\vec{v} = \begin{pmatrix} a_1 & a_2 & \dots & a_m \end{pmatrix}^T$ is a k-flow for an $n \times m$ matrix M if $M\vec{v} = 0$ and, for $1 \le i \le m$, a_i is an integer satisfying $0 < |a_i| < k$. The existence (or non-existence) of zero-sum k-flows for incidence matrices of combinatorial objects has been the object of much study.

Let M be the $\{\pm 1, 0\}$-incidence matrix of vertices versus arcs of a directed graph G. In other words, the rows and columns of M are indexed by the vertices and arcs of G, respectively, and the (i, j) entry of M is 1 if the i^{th} vertex is the head of the j^{th} directed edge, -1 if the i^{th} vertex is the tail of the j^{th} directed edge, and zero otherwise. The celebrated Four Color Theorem (Appel and Haken [5, 6] and Appel, Haken, and Koch [7]) is equivalent to the statement that if G is a bridge-less planar directed graph (a *bridge* a.k.a. a *cut-edge* is an edge whose removal increases the number of connected components of the graph) then M has a 4-flow (see Tutte [15] as well as Seymour [14]). Further, a famous conjecture of Tutte [15] asserts that *every* $\{\pm 1, 0\}$-incidence matrix of vertices versus arcs of a bridge-less directed graph has a 5-flow. The best result toward this conjecture is that of Seymour [13, 14] who proved that such matrices must have a 6-flow. Because of the connection to these

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major results, the literature on zero-sum flows on directed graphs is extensive. (For directed graphs, what we have called a k-flow is called a *nowhere-zero k-flow.*)

Now, let M be the $\{0,1\}$-incidence matrix of vertices versus edges of a (simple undirected) graph G. The rows and columns of M are indexed by the vertices and edges of G, respectively, and the (i,j) entry is 1 if the i^{th} vertex is on the j^{th} edge and 0 otherwise. A conjecture of Akbari, Ghareghani, Khosrovshahi, and Mahmoody [2] states that if M has a zero-sum flow, then M must have a 6-flow. The same authors also characterized the graphs whose incidence matrix does have a zero-sum flow. This conjecture turns out to be equivalent to an older conjecture of Bouchet [10] for bidirected graphs—for the equivalence of the two conjectures see Akbari et al [1]—and has been proved for bipartite graphs (Akbari et al [2]), and for r-regular graphs with $r \geq 3$ (Akbari et al [1, 2, 3] and Zare [18]).

Let $[v] = \{1, \dots, v\}$, and define a *k-subset* of $[v]$ as a subset of $[v]$ of size k. If $\mathcal{B}$ is a family of k-subsets of $[v]$, then $\mathcal{B}$ is called a t-(v,k,λ) design if every t-subset of $[v]$ is contained in exactly λ elements of $\mathcal{B}$. The elements of $[v]$ and $\mathcal{B}$ are called the *points* and the *blocks* of the design, respectively. The design is called *symmetric* if $v = |\mathcal{B}|$. Let M be the $\{0,1\}$-incidence matrix of points versus blocks of a t-(v,k,λ) design. Then Akbari, Khosrovshahi, and Mofidi [4] prove that M has a zero-sum flow if $t = 2$ and the design is non-symmetric. They also conjecture that for *any* non-symmetric t-(v,k,λ) design, M has a 5-flow, and, for $v > 7$, and every $2-(v,3,1)$ design (a.k.a. Steiner triple systems), M has a 3-flow. In the design-theory literature, a zero-sum flow is called a *nowhere-zero trade.*

For our final example of k-flows for incidence matrices of combinatorial objects, let $W_{tk}(v)$ be the incidence matrix of t-subsets versus k-subsets of $[v]$, with $1 \leq t \leq k \leq v$. In other words, the rows and columns of $W_{tk}(v)$ are indexed by the t-subsets and k-subsets of a set with v elements, and the (i,j) entry of this matrix is 1 if the i^{th} t-set is contained in the j^{th} k-set, and 0 otherwise. The family of *all* k-subsets of $[v]$ is a t-$(v,k,\binom{v-t}{k-t})$ design and so the results and conjectures for t-(v,k,λ) designs apply to it. In fact, Akbari, Khosrovshahi, and Mofidi [4] conjecture that, as long as $v \neq k+t$, $W_{tk}(v)$ has a 3-flow. They prove this conjecture for $t = 2$.

1.2. **Summary of Main Results.** We now turn to the linear lattices that are the object of this paper. Let q be a prime power and $\mathbb{F}_q$ the field with q elements. Let $\mathcal{L}_n(q)$ be the *linear lattice* (also known as the *subspace lattice*) of subspaces of the n-dimensional vector space $(\mathbb{F}_q)^n$ over the field of scalars $\mathbb{F}_q$, ordered by inclusion. For $0 \leq m \leq n$, *level* m of $\mathcal{L}_n(q)$ is the set of subspaces of dimension m of $(\mathbb{F}_q)^n$.

For $0 < \ell < m < n$, let $M = M_\ell^m$ be the incidence matrix of level ℓ versus level m of $\mathcal{L}_n(q)$. That is, the rows and columns of M are indexed by the elements of $\mathcal{L}_n(q)$ of dimension ℓ and m, respectively; and the (i,j) entry of M equals 1 if the i^{th} subspace of dimension ℓ is contained in the j^{th} subspace of dimension m, and 0 otherwise.

The rank number for level $0 \leq m \leq n$ of $\mathcal{L}_n(q)$ is given by the *q-binomial coefficient* $\begin{bmatrix} n \\ m \end{bmatrix}_q = \frac{[n]_q!}{[m]_q![n-m]_q!}$, where $[m]_q = (q^m - 1)/(q-1)$ if $m > 0$, with $[0]_q = 1$, and where $[m]_q! = \prod_{i=0}^m [i]_q$. Thus, $M = M_\ell^m$ is an $\begin{bmatrix} n \\ \ell \end{bmatrix}_q \times \begin{bmatrix} n \\ m \end{bmatrix}_q$ matrix. We consider M as a matrix over the real numbers. Also, because the linear lattice is unimodal and symmetric around its middle level(s), we will assume that $\ell < \min\{m, n-m\}$ so that M has a nontrivial nullspace $\mathcal{N}(M)$.

Our first result, that M admits a zero-sum flow (see Wilson [17] for a very different proof), follows from a more general statement about bipartite graphs. If G is a bipartite graph, we write $G = (\mathcal{A}, \mathcal{B}, E)$ where $\mathcal{A} \cup \mathcal{B}$ is the set of vertices of G, the set of edges is E, and all the edges have one end in $\mathcal{A}$ and one end in $\mathcal{B}$. For $G = (\mathcal{A}, \mathcal{B}, E)$, the incidence matrix of elements of $\mathcal{A}$ versus those of $\mathcal{B}$—also called the *biadjacency matrix* of G—has rows and columns indexed by $\mathcal{A}$ and $\mathcal{B}$ respectively, and with the (i, j) entry of the matrix equal to 1 if the i^{th} vertex in $\mathcal{A}$ is adjacent to the j^{th} vertex in $\mathcal{B}$.

Theorem 1. *Suppose $G = (\mathcal{A}, \mathcal{B}, E)$ is a bipartite graph. Suppose further that the automorphism group $\mathrm{Aut}(G)$ acts transitively on $\mathcal{B}$. Let M denote the incidence matrix of elements of $\mathcal{A}$ versus those of $\mathcal{B}$. If the nullspace of M is nontrivial, then M admits a zero-sum flow.*

As a vector space, $(\mathbb{F}_q)^n$ is isomorphic to the field $\mathbb{F}_{q^n}$ with q^n elements; we fix such an isomorphism to identify $\mathbb{F}_{q^n}$ with $(\mathbb{F}_q)^n$. But $\mathbb{F}_{q^n}$ also admits a multiplicative structure. In particular, the multiplicative subgroup of the field is cyclic: $\mathbb{F}_{q^n}^{\times} = \langle x \rangle$ for some $x \in \mathbb{F}_{q^n}^{\times}$. We take advantage of this additional multiplicative structure to prove our second result. (See Sarkis et al. [12] for another example of the use of this algebraic method for proving combinatorial results in the linear lattices.)

Theorem 2. *Suppose $n \geq 4$ and $2 \leq m \leq n-2$, and let $M = M_1^m$ be the incidence matrix of level 1 versus level m of the linear lattice $\mathcal{L}_n(q)$. If $m \mid n$, then M admits an $([m]_q + 1)$-flow. If $n - m \mid n$, then M admits an $([n-m]_q + 1)$-flow. If neither m nor $n - m$ divide n, then M admits a 2- or 3-flow.*

Many of our proofs will use the straightforward observation that a zero-sum flow of M_ℓ^m corresponds to a labeling of the m-dimensional subspaces of $\mathcal{L}_n(q)$ with nonzero numbers such that, for each ℓ-dimensional subspace $V \in \mathcal{L}_n(q)$, the sum of the labels of those m-dimensional subspaces that contain V equals zero. To illustrate, we end this section with a quick proof of a stronger version of Theorem 2 when $m = 2$. The proof uses known results about spreads and parallelisms.

A *spread* is a collection of 2-dimensional subspaces of $\mathcal{L}_n(q)$ such that every subspace of dimension 1 is contained in exactly one of the 2-dimensional subspaces. A *parallelism* or a *packing* is a partition of level 2 of $\mathcal{L}_n(q)$ into spreads. It is known that a parallelism exists if n is even and $q = 2$ (Baker [8], and Wettl [16] who gives a different construction) or if $n \geq 4$ is a power of 2 and q is an arbitrary prime power (Denniston [11] for $n = 4$ and Beutelspacher [9] for the general case).

Special Case of Theorem 2. *Suppose $n \geq 4$, and let $M = M_1^2$ be the incidence matrix of level 1 versus level 2 of the linear lattice $\mathcal{L}_n(q)$. If n is even and $q = 2$, or if n is a power of 2 and q is an arbitrary prime power, then M admits a 2- or 3-flow.*

Proof. Given a parallelism with an even number of spreads, assign $+1$ to each subspace in half of the spreads and -1 to the rest to get a 2-flow for M. If the number of spreads is odd, then first assign $+2$ to the subspaces in one spread and -1 to the subspaces in two other spreads. Complete a 3-flow for M by assigning $+1$ to the subspaces in half of the remaining spreads and -1 to the rest of the subspaces. $\square$

2. Bipartite graphs with high regularity

In this section, we prove Theorem 1 by associating flows with vertex labels, and by showing that the bipartite graph's automorphism group allows us to permute these labels sufficiently so that each vertex gets a nonzero label.

Lemma 3. *Let F be an infinite field. Suppose $V \subseteq F^n$ is a vector subspace with the property that for each $1 \leq i \leq n$, V contains a vector $\vec{v}_i$ whose i^{th} entry is nonzero. Then V contains a vector whose entries are all nonzero.*

Proof. We proceed by induction to show that, for each $1 \leq i \leq n$, there exists a vector $\vec{w}_i \in V$ whose first i entries are all nonzero; in that case $\vec{w}_n$ is the vector we seek. Clearly, $\vec{w}_1 = \vec{v}_1$ satisfies this property. Assume that for some $1 \leq i \leq n-1$, such a $\vec{w}_i \in V$ exists. Consider the set $\{\vec{w}_i + \alpha\vec{v}_{i+1} \mid \alpha \in F\}$. This is an infinite set. However, for each $1 \leq j \leq i+1$, there exists at most one $\alpha_j \in F$ such that the j^{th} entry of $\vec{w}_i + \alpha_j\vec{v}_{i+1}$ equals zero. Therefore, there exists infinitely many vectors of the form $\vec{w}_i + \alpha\vec{v}_{i+1}$ whose first $i+1$ entries are all nonzero. □

Proof of Theorem 1. Recall that the incidence matrix M has its rows indexed by $\mathcal{A}$ and its columns by $\mathcal{B}$. Let $|\mathcal{B}| = n$. Then the nullspace $\mathcal{N}(M)$ is a subspace of $\mathbb{Q}^n$. Given $\vec{v} \in \mathbb{Q}$ and $b \in \mathcal{B}$, let $\vec{v}(b) \in \mathbb{Q}$ be the entry of $\vec{v}$ indexed by b; that is, if b corresponds to the i^{th} column of M, then $\vec{v}(b)$ is the i^{th} entry of $\vec{v}$. Since M is a $\{0,1\}$-matrix, then

$$\vec{v} \in \mathcal{N}(M) \iff \text{for each } a \in \mathcal{A}, \sum_{(a,b)\in E} \vec{v}(b) = 0. \tag{1}$$

In other words, a vector in the nullspace of M corresonds to a labeling of the vertices in $\mathcal{B}$ such that, for each $a \in \mathcal{A}$, the sum of the labelings of vertices in $\mathcal{B}$ that are adjacent to a equals zero.

An automorphism $\varphi \in \mathrm{Aut}(G)$ is a permutation of $\mathcal{A}$ and of $\mathcal{B}$ such that $(a,b) \in E$ if and only if $(\varphi(a), \varphi(b)) \in E$. For $\varphi \in \mathrm{Aut}(G)$ and $\vec{v} \in \mathcal{N}(M)$, define $\vec{v}_\varphi \in \mathcal{N}(M)$ by $\vec{v}_\varphi(b) = \vec{v}(\varphi^{-1}(b))$. To verify that $\vec{v}_\varphi$ is indeed a nullspace vector, note that for each $a \in \mathcal{A}$,

$$\begin{aligned}\sum_{(a,b)\in E} \vec{v}_\varphi(b) &= \sum_{(a,b)\in E} \vec{v}(\varphi^{-1}(b)) \\ &= \sum_{(\varphi^{-1}(a),\varphi^{-1}(b))\in E} \vec{v}(\varphi^{-1}(b)) \\ &= 0\end{aligned}$$

The second equality follows from the fact that, since $\varphi^{-1} \in \mathrm{Aut}(G)$, then $(a,b) \in E$ if and only if $(\varphi^{-1}(a), \varphi^{-1}(b)) \in E$.

Since $\mathcal{N}(M)$ is nontrivial, there must exist $\vec{v} \in \mathcal{N}(M)$ and $b_1 \in \mathcal{B}$ such that $\vec{v}(b_1) \neq 0$. For an arbitrary $b_2 \in \mathcal{B}$, let $\varphi \in \mathrm{Aut}(G)$ such that $\varphi(b_1) = b_2$. Such a φ exists because $\mathrm{Aut}(G)$ acts transitively on $\mathcal{B}$. Thus $\vec{v}_\varphi$ has the property that $\vec{v}_\varphi(b_2) = \vec{v}(b_1) \neq 0$. By Lemma 3, the result follows. □

Corollary 4. *Suppose M_ℓ^m is the incidence matrix of level ℓ versus level m of the linear lattice $\mathcal{L}_n(q)$. If $\ell < \min\{m, n-m\}$, then M_ℓ^m admits a zero-sum flow.*

3. Orbits in $\mathcal{L}_n(q)$

Our proof of Theorem 2 relies on a group action on $\mathcal{L}_n(q)$ that we study in more detail next. We are in particular interested in the orbit sizes under this action.

Continue to denote by x a generator of the multiplicative group of the field $\mathbb{F}_{q^n}$. That is, x is an element of $\mathbb{F}_{q^n}$ of order $|x| = q^n - 1$. Additionally, x is a primitive element of the field extension $\mathbb{F}_{q^n}/\mathbb{F}_q$. Thus, x is the root of a monic irreducible polynomial $m_x(t) \in \mathbb{F}_q[t]$ of degree n. We fix an isomorphism $(\mathbb{F}_q)^n \cong \mathbb{F}_{q^n}$ and, by abuse of notation, we use x to denote as well the corresponding vector in $(\mathbb{F}_q)^n$.

Consider the action of $\mathbb{F}_{q^n}^\times = \langle x \rangle$ on $\mathcal{L}_n(q)$ defined as follows: if $V \in \mathcal{L}_n(q)$, then $x^i \cdot V = \{x^i \vec{v} \mid \vec{v} \in V\}$. It is straightforward to verify that this is indeed a group action, and that the action preserves rank.

Since $\mathbb{F}_q \subset \mathbb{F}_{q^n}$, some vectors are also scalars.

Lemma 5. *The vector x^i is a scalar if and only if $[n]_q \mid i$.*

Proof. We have $x^i \in \mathbb{F}_q \iff (x^i)^q = x^i \iff x^{i(q-1)} = 1 \iff |x| = q^n - 1 \mid i(q-1) \iff (q^n-1)/(q-1) \mid i$, as desired. □

Corollary 6. *The vectors x^i and x^j are scalar multiples of each other if and only if $[n]_q \mid i - j$.*

Corollary 7. *The 1-dimensional subspaces $\operatorname{span}\{x^i\}$ and $\operatorname{span}\{x^j\}$ are equal if and only if $[n]_q \mid i - j$. In particular, the 1-dimensional subspaces of $\mathcal{L}_n(q)$ are given by $\operatorname{span}\{x^i\}$ for $0 \le i \le [n]_q - 1$.*

Example. *Suppose $n = 4$ and $q = 3$. The polynomial $t^4 + t + 2 \in \mathbb{F}_3[t]$ is irreducible. Suppose x is one of its roots. Then $\mathbb{F}_3[x] \cong \mathbb{F}_{3^4}$, and $\mathbb{F}_{3^4}^\times = \langle x \rangle$. Noting that $\{1, x, x^2, x^3\}$ forms a basis for $\mathbb{F}_3[x]$ over $\mathbb{F}_3$, consider the isomorphism $\mathbb{F}_3[x] \to (\mathbb{F}_3)^4$ given by $x^{i-1} \mapsto e_i$, the i^{th} standard basis vector, where $1 \le i \le 4$.*

The action of multiplication by x on $(\mathbb{F}_3)^4$ is a linear transformation whose matrix representation in the coordinates of the standard basis is

$$X = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

Following Lemma 5, we see that $X^{[4]_3} = X^{40} = 2I_4$ and $X^{3^4-1} = I_4$. Note also that the characteristic polynomial of X is $t^4 + t + 2$, and that $\det(X) = 2$, the field-theoretic norm of x over $\mathbb{F}_3$.

The action of $\langle x \rangle$ on $\mathcal{L}_4(3)$ can now be computed in one of two ways: either by writing all nonzero vectors as powers of x, or by multiplying coordinatized vectors by X. For instance, if $V = \operatorname{span}\{\begin{pmatrix}1 & 1 & 0 & 1\end{pmatrix}^T\}$ then $x \cdot V = \operatorname{span}\{\begin{pmatrix}1 & 0 & 1 & 0\end{pmatrix}^T\}$.

For $V \in \mathcal{L}_n(q)$, denote by $\mathcal{O}_V$ the orbit of V under this action.

Lemma 8. *For $V \in \mathcal{L}_n(q)$, $\mathbb{F}_q^\times \subseteq \operatorname{stab}_{\langle x \rangle}(V)$, and so $|\mathcal{O}_V| \mid [n]_q$.*

Proof. Clearly, if $a \in \mathbb{F}_q$ then $a \cdot V = V$. Thus $|\mathcal{O}_V| = |\langle x \rangle|/|\operatorname{stab}\langle x \rangle(V)| = \dfrac{q^n - 1}{k(q-1)}$, where $k = |\operatorname{stab}_{\langle x \rangle}(V) : \mathbb{F}_q^\times|$. □

Corollary 7 shows that, restricted to the 1-dimensional subspaces of $\mathcal{L}_n(q)$, the action of multiplication by x is transitive. Hence, if $\dim(V) = 1$ then $|\mathcal{O}_V| = [n]_q$.

Lemma 8 shows that $[n]_q$ is the largest possible orbit size. We continue to explore the allowable values of $|\mathcal{O}_V|$.

Lemma 9. *Suppose $V \in \mathcal{L}_n(q)$ is a subspace of dimension $\ell > 0$, and $|\mathcal{O}_V| = d$. Then $\left[\mathbb{F}_q(x^d) : \mathbb{F}_q\right] \leq \ell$.*

Proof. Suppose $\{x^{b_1}, \cdots, x^{b_\ell}\}$ form a basis for V, where $b_1, \cdots, b_\ell$ are integers. Then $\{x^{b_1+d}, \cdots, x^{b_\ell+d}\}$ form a basis for $x^d \cdot V$. Since $V = x^d \cdot V$, there must exist an $\ell \times \ell$ matrix A with entries in $\mathbb{F}_q$ such that

$$A \begin{pmatrix} x^{b_1} \\ \vdots \\ x^{b_\ell} \end{pmatrix} = \begin{pmatrix} x^{b_1+d} \\ \vdots \\ x^{b_\ell+d} \end{pmatrix} = x^d \begin{pmatrix} x^{b_1} \\ \vdots \\ x^{b_\ell} \end{pmatrix}$$

In other words, x^d is an eigenvalue of a A, and hence the root of a polynomial over $\mathbb{F}_q$ of degree ℓ. □

Corollary 10. *Suppose $V \in \mathcal{L}_n(q)$ is a subspace of dimension $\ell > 0$ such that ℓ is smaller than the smallest divisor of n other than 1. Then $|\mathcal{O}_V| = [n]_q$. In particular, if n is prime, then $|\mathcal{O}_V| = [n]_q$ for all $V \neq \{0\}, (\mathbb{F}_q)^n$.*

Proof. If $|\mathcal{O}_V| = d$ then $\left[\mathbb{F}_q(x^d) : \mathbb{F}_q\right]$ must be a proper factor of n, and so it must equal 1. Thus $x^d \in \mathbb{F}_q$, and so by Lemma 5, $[n]_q \mid d$. □

Suppose that $r \mid n$ and $V \in \mathcal{L}_{\frac{n}{r}}(q^r)$. Thus V is a vector space over $\mathbb{F}_{q^r}$ of dimension at most n/r. Since $\mathbb{F}_{q^r}$ is itself a vector space of dimension r over $\mathbb{F}_q$, then V is a vector space over $\mathbb{F}_q$ of dimension at most n. By identifying both the $\frac{n}{r}$-dimensional subspace of $\mathcal{L}_{\frac{n}{r}}(q^r)$ and the n-dimensional subspace of $\mathcal{L}_n(q)$ with $\mathbb{F}_{q^n}$, we get a natural embedding $\mathcal{L}_{\frac{n}{r}}(q^r) \subset \mathcal{L}_n(q)$. In that case, the action of $\mathbb{F}_{q^n}^\times$ restricts naturally to $\mathcal{L}_{\frac{n}{r}}(q^r)$, so that if $V \in \mathcal{L}_{\frac{n}{r}}(q^r)$ then $\mathcal{O}_V \subset \mathcal{L}_{\frac{n}{r}}(q^r)$ as well; moreover, by Lemma 8, $|\mathcal{O}_V| \mid [\frac{n}{r}]_{q^r}$.

Proposition 11. *Suppose $V \in \mathcal{L}_n(q)$ is an m-dimensional subspace. Let $d = |\mathcal{O}_V|$ and $\mathbb{F}_q(x^d) = \mathbb{F}_{q^r}$ for some $r \mid n$. Then $V = \oplus_{i=1}^k x^{a_i}\mathbb{F}_{q^r}$ for some $0 \leq a_1, \cdots, a_k \leq [n]_q - 1$ such that $m = kr$. In particular, V is a k-dimensional subspace of $\mathcal{L}_{\frac{n}{r}}(q^r)$. Additionally, $|\mathcal{O}_V| = [n]_q/[r]_q = [\frac{n}{r}]_{q^r}$.*

Proof. Let $\{x^{a_1}, \cdots, x^{a_m}\}$ be a basis for V over $\mathbb{F}_q$ for some $0 \leq a_1, \cdots, a_m \leq [n]_q - 1$. Since $x^d \cdot V = V$, then for each $1 \leq i \leq m$ and each $j \in \mathbb{Z}$, we have $x^{a_i}x^{jd} \in V$. Thus, $x^{a_i}\mathbb{F}_{q^r} \subseteq V$ as well, since $\mathbb{F}_{q^r} = \mathbb{F}_q(x^d)$ is spanned by $\{x^{jd} \mid j \in \mathbb{Z}\}$. Clearly, $V \subseteq \sum_{i=1}^m x^{a_i}\mathbb{F}_{q^r}$ because $\{x^{a_1}, \cdots, x^{a_m}\}$ span V, so in fact $V = \sum_{i=1}^m x^{a_i}\mathbb{F}_{q^r}$. Note that $x^{a_i}\mathbb{F}_{q^r}^\times$ are cosets of $\mathbb{F}_{q^r}^\times$ in $\mathbb{F}_{q^n}^\times$, and so they either are distinct or coincide. By reordering if necessary, suppose without loss of generality that $a_1, \cdots, a_k$ are representatives of the distinct cosets among $\{a_i\mathbb{F}_{q^r}^\times \mid 1 \leq i \leq m\}$. Then $V = \oplus_{i=1}^k x^{a_i}\mathbb{F}_{q^r} \in \mathcal{L}_{\frac{n}{r}}(q^r)$, as desired. To prove the final claim, first note that $\mathbb{F}_{q^r}^\times \subseteq \mathrm{stab}_{\langle x \rangle}(V)$ by Lemma 8. Also, if $x^b \in \mathrm{stab}_{\langle x \rangle}(V)$, then so is x^e, where e is the remainder of b upon division by d. Since $e < d$ and $x^e \cdot V = V$, we conclude that $e = 0$, and so $\mathrm{stab}_{\langle x \rangle}(V) = \mathbb{F}_{q^r}^\times$. □

Corollary 12. *For $0 \leq m \leq n$, there exists an m-dimensional subspace $V \in \mathcal{L}_n(q)$ with $|\mathcal{O}_V| < [n]_q$ if and only if $\gcd(m, n) > 1$. In that case $V \in \mathcal{L}_{\frac{n}{r}}(q^r)$ for some $1 < r \mid \gcd(m, n)$.*

Proof. If $|\mathcal{O}_V| = d < [n]_q$, then by Lemma 5, $\mathbb{F}_q(x^d) = \mathbb{F}_{q^r} \supsetneq \mathbb{F}_q$. In that case, Proposition 11 implies that $1 < r \mid \gcd(m,n)$. Conversely, if $\gcd(m,n) > 1$, let $r = \gcd(m,n)$, $k = m/r$ and $V = \oplus_{i=1}^{k} x^i \mathbb{F}_{q^r}$; then $V \in \mathcal{L}_{\frac{n}{r}}(q^r)$ and $|\mathcal{O}_V| \mid [\frac{n}{r}]_{q^r}$ by Lemma 8. $\square$

We end this section by determining when an orbit is the only one of its size on a given level. The result will be useful in the proof of the first part of Theorem 2.

Lemma 13. $\dfrac{\begin{bmatrix} \frac{n}{2} \\ \frac{m}{2} \end{bmatrix}_{q^2}}{\begin{bmatrix} n \\ m \end{bmatrix}_q} < q^{\frac{m}{2}(m-n)}$.

Proof. For any $k > 0$, $\dfrac{[\frac{k}{2}]_{q^2}}{[k]_q} = \dfrac{\frac{(q^2)^{\frac{k}{2}}-1}{q^2-1}}{\frac{q^k-1}{q-1}} = \dfrac{1}{q+1}$. Therefore,

$$\frac{[\frac{k}{2}]_{q^2}!}{[k]_q!} = \prod_{i=0}^{\frac{k}{2}} \frac{[\frac{k-2i}{2}]_{q^2}}{[k-2i]_q} \prod_{j=0}^{\frac{k}{2}-1} \frac{1}{[k-(2j+1)]_q} = \frac{1}{(q+1)^{\frac{k}{2}}} \prod_{j=0}^{\frac{k}{2}-1} \frac{1}{[k-(2j+1)]_q}$$

Substituting, we get

$$\begin{aligned} \frac{\begin{bmatrix} \frac{n}{2} \\ \frac{m}{2} \end{bmatrix}_{q^2}}{\begin{bmatrix} n \\ m \end{bmatrix}_q} &= \frac{\left(\frac{1}{(q+1)^{\frac{n}{2}}} \prod_{j=0}^{\frac{n}{2}-1} \frac{1}{[n-(2j+1)]_q}\right)}{\left(\frac{1}{(q+1)^{\frac{m}{2}}} \prod_{j=0}^{\frac{m}{2}-1} \frac{1}{[m-(2j+1)]_q}\right)\left(\frac{1}{(q+1)^{\frac{n-m}{2}}} \prod_{j=0}^{\frac{n-m}{2}-1} \frac{1}{[(n-m)-(2j+1)]_q}\right)} \\ &= \frac{\left(\prod_{j=0}^{\frac{m}{2}-1} \frac{1}{[n-(2j+1)]_q}\right)}{\left(\prod_{j=0}^{\frac{m}{2}-1} \frac{1}{[m-(2j+1)]_q}\right)} = \prod_{j=0}^{\frac{m}{2}-1} \frac{q^{m-(2j+1)}-1}{q^{n-(2j+1)}-1} < q^{\frac{m}{2}(m-n)} \end{aligned}$$

$\square$

Lemma 14. *If $n = 4$, then there are q orbits of size $[4]_q$ at level 2 of $\mathcal{L}_4(q)$. If $n \geq 5$ and $2 \leq m \leq n-2$, then there are at least 5 orbits of size $[n]_q$ at level m.*

Proof. Proposition 11 shows that there is exactly one orbit of size $[2]_{q^2}$ at level 2 of $\mathcal{L}_4(q)$, while the rest have size $[4]_q$. There are $\begin{bmatrix} 4 \\ 2 \end{bmatrix}_q$ total subspaces at level 2, and so the number of 2-dimensional subspaces in $\mathcal{L}_4(q)$ whose orbits have size $[4]_q$ equals

$$\begin{bmatrix} 4 \\ 2 \end{bmatrix}_q - [2]_{q^2} = \frac{[4]_q[3]_q}{[2]_q} - [2]_{q^2} = ([3]_q - 1)[2]_{q^2} = q(q+1)(q^2+1) = q[4]_q$$

If $n = 5$ then the result follows by Corollary 10, since all nontrivial orbits have size $[5]_q$, and the number of such orbits at level $m = 2, 3$ equals $\begin{bmatrix} 5 \\ 2 \end{bmatrix}_q / [5]_q = q^2+1 \geq 5$.

Suppose $n \geq 6$. By Corollary 12, every subspace $V \in \mathcal{L}_n(q)$ of dimension m with orbit size $|\mathcal{O}_V| < [n]_q$ is in fact a subspace of $\mathcal{L}_{\frac{n}{r}}(q^r)$ for some divisor $1 < r \mid (m,n)$. Our proof will rely on an upper bound for the proportion of such subspaces with "small" orbits.

Note that if $r \mid s \mid n$ then $\mathcal{L}_{\frac{n}{s}}(q^s) \subseteq \mathcal{L}_{\frac{n}{r}}(q^r) \subseteq \mathcal{L}_n(q)$. So for counting purposes, we need only consider the prime divisors of (m,n) in the computation that follows.

Applying Lemma 13, we see that the proportion of subspaces at level m whose orbits have size less than $[n]_q$ is

$$\frac{\sum_{r|(m,n),r\text{ prime}}\left[\begin{smallmatrix}\frac{n}{r}\\ \frac{m}{r}\end{smallmatrix}\right]_{q^r}}{\left[\begin{smallmatrix}n\\ m\end{smallmatrix}\right]_q} \leq \frac{\log_2(n)\left[\begin{smallmatrix}\frac{n}{2}\\ \frac{m}{2}\end{smallmatrix}\right]_{q^2}}{\left[\begin{smallmatrix}n\\ m\end{smallmatrix}\right]_q} < \log_2(n)q^{\frac{m}{2}(m-n)} \leq \log_2(n)q^{2-n} < \frac{1}{4}$$

Since $\left[\begin{smallmatrix}n\\ m\end{smallmatrix}\right]_q \geq \left[\begin{smallmatrix}n\\ 2\end{smallmatrix}\right]_q = [n]_q\frac{[n-1]_q}{[2]_q} \geq [n]_q\frac{[5]_2}{[2]_2} > 10[n]_q$, then there are more than $\frac{3}{4}\cdot 10[n]_q = 7.5[n]_q$ subspaces at level m with orbit size $[n]_q$. □

Suppose $V \in \mathcal{L}_n(q)$ is an m-dimensional subspace. Call $\mathcal{O}_V$ *lonely* if no other orbit at level m of $\mathcal{L}_n(q)$ has size $|\mathcal{O}_V|$. The orbits at levels 1 and $n-1$ are lonely by Corollary 12. The next result shows that, in a sense, these are the only lonely orbits.

Proposition 15. *Suppose $n \geq 4$ and $2 \leq m \leq n-2$. Then there is precisely one lonely orbit at level m of $\mathcal{L}_n(q)$ if and only if either m nor $n-m$ divides n. In that case, the lonely orbit is of size $[\frac{n}{m}]_{q^m}$ (if $m \mid n$) or $[\frac{n}{n-m}]_{q^{n-m}}$ (if $n-m \mid n$).*

Proof. Suppose that $m \mid n$. Then the subspace $V = \mathbb{F}_{q^m}$ is m-dimensional over $\mathbb{F}_q$, and $\mathcal{O}_V$ is the only orbit of size $[\frac{n}{m}]_{q^m}$ at level m of $\mathcal{L}_n(q)$ because $\mathcal{O}_V$ is the only orbit at level 1 of $\mathcal{L}_{\frac{n}{m}}(q^m)$. Similarly, if $m' = n-m \mid n$ and $V' = \mathbb{F}_{q^{m'}}$, then $\mathcal{O}_{V'}$ is the only orbit at level m' of orbit size $[\frac{n}{m'}]_{q^{m'}}$. By Proposition 11, there is precisely one orbit $\mathcal{O}_V$ at level m of $\mathcal{L}_n(q)$ of size $[\frac{n}{m'}]_{q^{m'}}$, namely, the orbit at level $(\frac{n}{m'}-1)$ of $\mathcal{L}_{\frac{n}{m'}}(q^{m'})$.

To prove that $\mathcal{O}_V$ is the only lonely orbit at level m, we note from Proposition 11 that if $W \notin \mathcal{O}_V$ is any other m-dimensional subspace, then $W \in \mathcal{L}_{\frac{n}{r}}(q^r)$ for some r that is a proper divisor of m and $n-m$. In that case, $|\mathcal{O}_W| = [\frac{n}{r}]_{q^r}$. Additionally, since either $\frac{n}{m} > 1$ or $\frac{n}{n-m} > 1$ is a proper divisor of $\frac{n}{r}$, we must have $\frac{n}{r} \geq 4$. Then by Lemma 14, $\mathcal{O}_W$ is not the only orbit at level $\frac{m}{r}$ of $\mathcal{L}_{\frac{n}{r}}(q^r)$ of size $[\frac{n}{r}]_{q^r}$.

Suppose that neither m nor $n-m$ divide n. If $n=4$ then the result is vacuously true, and if $n=5$ then the result follows from Corollary 12, since all orbits would be of size $[5]_q$. Also, if $n \geq 5$ and $V \in \mathcal{L}_n(q)$ is a subspace of dimension m with $|\mathcal{O}_V| = [n]_q$, then Lemma 14 implies $\mathcal{O}_V$ is not lonely. We proceed by induction on n to prove the result for the case $|\mathcal{O}_V| < [n]_q$. Suppose for some $n \geq 6$ the result is true for $\mathcal{L}_4(q), \cdots, \mathcal{L}_{n-1}(q)$. If $|\mathcal{O}_V| < [n]_q$, then V is an $\frac{m}{r}$-dimensional subspace of $\mathcal{L}\frac{n}{r}(q^r)$ for some $1 < r \mid (m,n)$ by Proposition 11. Note that neither $\frac{m}{r}$ nor $\frac{n}{r} - \frac{m}{r}$ divides $\frac{n}{r}$. This necessarily implies $\frac{n}{r} \geq 5$. So by the inductive hypothesis, $\mathcal{O}_V$ is not the only orbit of size $|\mathcal{O}_V|$ in $\mathcal{L}_{\frac{n}{r}}(q^r)$.

□

4. The incidence matrix M_1^m

In this section, we show that the incidence matrix M_1^m between levels 1 and m of the linear lattice $\mathcal{L}_n(q)$ admits a zero-sum 2- or 3-flow if neither m nor $n-m$ divides n. In case m or $n-m$ divides n, then M_1^m admits an $([m]_q+1)$-flow or $([n-m]_q+1)$-flow, respectively.

For an ℓ-dimensional subspace $V \in \mathcal{L}_n(q)$, define the *shadow of V at level i* by $\triangle_i(V) = \{U \in \mathcal{L}_n(q) \mid \dim(U) = i \text{ and } U \subseteq V\}$, and the *(total) shadow of V* by $\triangle(V) = \cup_{i\leq\ell}\triangle_i(V)$. Similarly, define the *shade of V at level i* by $\nabla_i(V) =$

$\{W \in \mathcal{L}_n(q) \mid \dim(W) = i \text{ and } V \subseteq W\}$, and the *(total) shade of* V by $\nabla(V) = \cup_{i \geq \ell} \nabla_i(V)$.

As in the proof of Theorem 1, it will be convenient to think of vectors in the nullspace $\mathcal{N}(M)$ as labelings of the subspaces of $\mathcal{L}_n(q)$ of dimension m such that, for each subspace $V \in \mathcal{L}_n(q)$ of dimension 1, the sum of the labelings of all subspaces of $\nabla_m(V)$ equals zero.

Lemma 16. *Suppose* $V, W \in \mathcal{L}_n(q)$. *Then* $|\nabla(V) \cap \mathcal{O}_W| = |\nabla(x^i \cdot V) \cap \mathcal{O}_W|$ *and* $|\triangle(W) \cap \mathcal{O}_V| = |\triangle(x^i \cdot W) \cap \mathcal{O}_V|$ *for all* i.

Proof. This follows directly from the fact that $V \subset W \iff x^i \cdot V \subset x^i \cdot W$. □

The previous Lemma asserts that every element of $\mathcal{O}_V$ is contained in the same number of elements of $\mathcal{O}_W$, and conversely every element of $\mathcal{O}_W$ contains the same number of elements of $\mathcal{O}_V$. So we define these numbers to be, respectively, *the incidence number of* $\mathcal{O}_V$ *to* $\mathcal{O}_W$ and *the incidence number of* $\mathcal{O}_W$ *to* $\mathcal{O}_V$, and denote them $|\mathcal{O}_V : \mathcal{O}_W|$ and $|\mathcal{O}_W : \mathcal{O}_V|$.

Corollary 17. *Suppose* $V, W \in \mathcal{L}_n(q)$. *If* $|\mathcal{O}_V : \mathcal{O}_W| > 0$ *then* $|\mathcal{O}_V|/|\mathcal{O}_W| = |\mathcal{O}_W : \mathcal{O}_V|/|\mathcal{O}_V : \mathcal{O}_W|$.

Proof. Since each element of $\mathcal{O}_V$ is contained in $|\mathcal{O}_V : \mathcal{O}_W|$ elements of $\mathcal{O}_W$, there are in total $|\mathcal{O}_V| \cdot |\mathcal{O}_V : \mathcal{O}_W|$ containments from $\mathcal{O}_V$ to $\mathcal{O}_W$. Similarly, there are $|\mathcal{O}_W| \cdot |\mathcal{O}_W : \mathcal{O}_V|$ containments from $\mathcal{O}_W$ to $\mathcal{O}_V$. Clearly, these numbers should be equal. □

Corollary 18. *Suppose* $V, W \in \mathcal{L}_n(q)$ *with* $1 = \dim(V) < \dim(W) = m$, *and* $\mathcal{O}_W$ *is a lonely orbit. If* $m \mid n$ *then* $|\mathcal{O}_V : \mathcal{O}_W| = 1$, *and if* $n - m \mid n$ *then* $|\mathcal{O}_V : \mathcal{O}_W| = \frac{[m]_q}{[n-m]_q}$.

Proof. Note that $|\mathcal{O}_V| = [n]_q$ by Corollary 7, and $|\mathcal{O}_W : \mathcal{O}_V| = \begin{bmatrix} m \\ 1 \end{bmatrix}_q = [m]_q$. Thus $|\mathcal{O}_V : \mathcal{O}_W| = |\mathcal{O}_W : \mathcal{O}_V| \cdot |\mathcal{O}_W|/|\mathcal{O}_V| = |\mathcal{O}_W| \frac{[m]_q}{[n]_q}$. The result now follows from Proposition 15. □

Corollary 19. *Suppose* $n \geq 5$, $V \in \mathcal{L}_n(q)$ *with* $\dim(V) = 1$, *and* $2 \leq m \leq n-2$. *Then there exist at least* 5 *distinct orbits at level* m *such that the incidence number of* $\mathcal{O}_V$ *to each of these orbits equals* $[m]_q$.

Proof. By Lemma 14, there are at least 5 orbits on level m of size $[n]_q$ each. If $\mathcal{O}_W$ is one such orbit, then Corollary 17 implies $|\mathcal{O}_V : \mathcal{O}_W| = |\mathcal{O}_W : \mathcal{O}_V| = [m]_q$. □

In light of the uniformity of the incidence degrees between orbits, it will be useful to consider an incidence matrix of orbits instead of subspaces. Given levels ℓ and m of $\mathcal{L}_n(q)$ with $0 < \ell < m < n$, consider the matrix $\widehat{M} = \widehat{M}_\ell^m$ whose rows are indexed by the distinct orbits of subspaces of dimension ℓ, and whose columns are indexed by the distinct orbits of subspaces of dimension m. The entry in $\widehat{M}$ corresponding to row $\mathcal{O}_V$ and column $\mathcal{O}_W$ equals $|\mathcal{O}_V : \mathcal{O}_W|$. We will call $\widehat{M}$ the *orbit incidence matrix* from level ℓ to m.

Lemma 20. *Suppose* $M = M_\ell^m$ *and* $\widehat{M} = \widehat{M}_\ell^m$ *are the incidence and orbit incidence matrices, respectively, from level* ℓ *to* m *of* $\mathcal{L}_n(q)$. *If* $\widehat{M}$ *has a* k*-flow for some integer* $k > 1$, *then so does* M.

Proof. Recall that we can think of a vector in the nullspace of M as a labeling of the subspaces of dimension m in $\mathcal{L}_n(q)$ such that, for each subspace $V \in \mathcal{L}_n(q)$ of dimension ℓ, the sum of labels of all dimension-m subspaces in ∇V equals zero. Let $W_1, \cdots, W_s$ be representatives of the distinct orbits of level m. Suppose $\vec{w} = \begin{pmatrix} w_1 & \cdots & w_s \end{pmatrix}^T$ is in the nullspace of $\widehat{M}$. If $W \in \mathcal{O}_{W_i}$, assign to W the label w_i. The proof will be complete when we show that this labeling corresponds to a vector in the nullspace of M. Suppose $V \in \mathcal{L}_n(q)$ is a subspace of dimension ℓ. By Lemma 16, for each $1 \leq i \leq s$, the sum of the labels in $\nabla V \cap \mathcal{O}_{W_i}$ equals $|\mathcal{O}_V : \mathcal{O}_{W_i}| w_i$. Thus, the sum of labels of all dimension-m subspaces in ∇V equals $\sum_i |\mathcal{O}_V : \mathcal{O}_{W_i}| w_i$. However, $\sum_i |\mathcal{O}_V : \mathcal{O}_{W_i}| w_i$ is also the dot product of $\vec{w}$ with the row of $\widehat{M}$ indexed by $\mathcal{O}_V$, and so it equals 0. □

Lemma 21. *Suppose A is a $1 \times s$ matrix with the property that if a is an entry of A, then A has more than one entry that equals a. Then A has a 2- or 3-flow.*

Proof. Suppose without loss of generality that $A = \begin{pmatrix} A_1 & A_2 & \cdots & A_\ell \end{pmatrix}$, where for each $1 \leq i \leq \ell$, $A_i = \begin{pmatrix} a_i & \cdots & a_i \end{pmatrix}$ is a $1 \times s_i$ matrix with $s_i > 1$ and $a_i \in \mathbb{R}$. For each i, construct a $1 \times s_i$ vector $\vec{v}_i$ as follows. If s_i is even, let $\vec{v}_i$ be a vector with $s_i/2$ entries equal 1 and the remaining $s_i/2$ entries equal -1. If s_i is odd, let $\vec{v}_i$ be a vector with one entry equal 2, $(s_i - 3)/2$ entries equal 1, and the remaining $(s_i + 1)/2$ entries equal -1. Then $\begin{pmatrix} \vec{v}_1 & \vec{v}_2 & \cdots & \vec{v}_\ell \end{pmatrix}^T$ is in the nullspace of A. □

Corollary 22. *Suppose A is a $1 \times s$ matrix with positive integer entries and $s \geq 5$ such the smallest entry of A appears exactly once, the largest entry appears with a multiplicity other than 2, and each of the remaining entries appears with multiplicity at least 2. Suppose also that the smallest entry divides the largest entry. Then A admits a $(k+1)$-flow, where k is the ratio of the largest to smallest entry of A.*

Proof. Write $A = \begin{pmatrix} a_1 & a_2 & \cdots & a_s \end{pmatrix}$, and suppose without loss of generality that $a_1 \geq a_2 \geq a_3 \geq \cdots > a_s$. Then $\begin{pmatrix} a_2 & a_3 & \cdots & a_{s-1} \end{pmatrix}$ satisfies the hypothesis of Lemma 21, and so it admits a 2- or 3-flow $\begin{pmatrix} y_2 & y_3 & \cdots & y_{s-1} \end{pmatrix}^T$. Therefore, $\begin{pmatrix} -1 & y_2 & y_3 & \cdots & y_{s-1} & \frac{a_1}{a_s} \end{pmatrix}^T$ is an $(\frac{a_1}{a_s} + 1)$-flow of A. □

Proof of Theorem 2. Let $M = M_1^m$, where $n \geq 4$ and $2 \leq m \leq n-2$. Then $\widehat{M} = \widehat{M_1^m}$ is a $1 \times s$ matrix by Corollary 7. By Lemma 20, any k-flow of $\widehat{M}$ can be extended to a k-flow of M, so it is sufficient to prove the results for $\widehat{M}$.

Suppose m or $n-m$ divides n. If $n = 4$ and $q = 2$, then $\widehat{M} = \begin{pmatrix} [2]_2 & [2]_2 & 1 \end{pmatrix}$, and so $\begin{pmatrix} 1 & -2 & [2]_2 \end{pmatrix}^T$ is a $([2]_2+1)$-flow of $\widehat{M}$. If $n = 4$ and $q > 2$, or if $n \geq 5$, then Lemma 14 and Corollary 19 imply that the largest entry of $\widehat{M}$ is $[m]_q$, and that entry appears with multiplicity at least 3. Additionally, Proposition 15 and Corollary 18 imply that the smallest entry of $\widehat{M}$ is 1 (if $m \mid n$) or $\frac{[m]_q}{[n-m]_q}$ (if $n - m \mid n$), and that entry appears with multiplicity 1. Finally, Proposition 15 implies that each of the remaining entries of $\widehat{M}$ has multiplicity at least 2. In other words, $\widehat{M}$ satisfies the hypothesis of Corollary 22, and hence admits an $([m]_q + 1)$-flow (if $m \mid n$) or an $([n-m]_q + 1)$-flow (if $n - m \mid n$).

If neither m nor $n-m$ divide n, then by Proposition 15, $\widehat{M}$ satisfies the hypothesis of Lemma 21, and so admits a zero-sum 2- or 3-flow. □

The orbit-based method described in the paper does not preclude a 2- or 3-flow for M_1^m in the case where m or $n-m$ divides n. Given the highly symmetric structure of the linear lattice, we conclude with the conjecture that M_1^m must have a 2- or 3-flow for all $2 \leq m \leq n-2$.

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